Proofs for “AGM-Style Revision of Beliefs and Intentions” (ECAI2016)

Abstract. We prove soundness and completeness for the logic PAL-P (Parameterized-time Action Logic with extended Preconditions), several propositions and lemmas, and representation theorems for the revision of beliefs and intentions. Please see the original ECAI submission for an explanation of the logic.

1 Completeness Proofs

Theorem 1 (Completeness Theorem). *The logic PAL-P is sound and strongly complete, i.e. $T \vdash \varphi$ iff $T \models \varphi$.*

Proof. $T \vdash \varphi \Rightarrow T \models \varphi$ can be proven by standard techniques.

Strongly completeness: $T \models \varphi \;\Rightarrow\; T \vdash \varphi$.

We prove strongly completeness by constructing a canonical model, but before this we introduce some concepts that we will need in different parts of the proof. These concepts will be largely familiar to most readers.

Definition 1 (Maximally consistent set (mcs)). *Given the logic PAL-P, a set of formulas T is PAL-P-consistent if one cannot derive a contradiction from it, i.e. if $\bot$ cannot be inferred from it, in the proof system for PAL-P. A set of formulas T^* is a* maximally PAL-P-consistent set *(mcs) if it is PAL-P-consistent and for every formula φ, either φ belong to the set or $\neg\varphi$ does.*
We denote the part of a mcs up to and include time t with T^*_t, formally: $T^*_t = T^* \cap Past(t)$ (see Def. 1 of original paper).

Lemma 1 (Lindenbaum's lemma). *Every consistent set of formulas can be extended to a maximal consistent set of formulas.*

Lemma 2 (The Deduction Theorem). $\Sigma \cup \{\varphi\} \vdash \psi \;\Rightarrow\; \Sigma \vdash \varphi \rightarrow \psi$

Definition 2 (Mcs Equivalence Relation). *Suppose some $t \in \mathbb{N}$ and two mcs's T^* and $\overline{T}^*$, we define the* equivalence relation *between T^* and $\overline{T}^*$, denoted by $T^* \equiv_t \overline{T}^*$ as follows: $T^* \equiv_t \overline{T}^*$ iff $T^* \cap Past(t) = \overline{T}^* \cap Past(t)$.*

Definition 3 (Equivalence class). *Let T^* be a mcs. $[T^*]_t$ is the set of all mcs's that are equivalent to T^* up and including time t, i.e. $[T^*]_t = \{\overline{T}^* \mid T^* \equiv_t \overline{T}^*\}$.*

The next step is to reduce truth of a formula in a maximal consistent set to membership of that set, which is the content of the truth lemma. We first present a lemma that we need in the proof of the valuation lemma, which follows after that.

Lemma 3. *Let $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$ be some set of PAL-P-formulas and abbreviate $\{\Box_t\varphi_1, \ldots, \Box_t\varphi_n\}$ with $\Box_t\Sigma$. If $\Sigma \vdash \varphi$, then $\Box_t\Sigma \vdash \Box_t\varphi$*

Proof. Suppose $\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi$. By the deduction lemma, $\vdash (\varphi_1 \wedge \ldots \wedge \varphi_n) \rightarrow \varphi$. Applying necessitation gives $\vdash \Box_t((\varphi_1 \wedge \ldots \wedge \varphi_n) \rightarrow \varphi)$, and from the K-axiom it follows that $\vdash \Box_t(\varphi_1 \wedge \ldots \wedge \varphi_n) \rightarrow \Box_t\varphi$. Since $\Box_t(\varphi_1 \wedge \ldots \wedge \varphi_n) \equiv \Box_t\varphi_1 \wedge \ldots \Box_t\varphi_n$, we obtain (1) $\vdash (\Box_t\varphi_1 \wedge \ldots \Box_t\varphi_n) \rightarrow \Box_t\varphi$. Finally, since (2) $\{\Box_t\varphi_1, \ldots, \Box_t\varphi_n\} \vdash \Box_t\varphi_1 \wedge \ldots \wedge \Box_t\varphi_n$ holds as well, we combine (1) and (2) and conclude that $\{\Box_t\varphi_1, \ldots, \Box_t\varphi_n\} \vdash \Box_t\varphi$.

Lemma 4. $T^*_t \vdash \Box_t T^*_t$

Proof. We show that for all $\varphi \in T^*_t$ we have $T^*_t \vdash \Box_t\varphi$ by induction on the depth of the proof. Take an arbitrary $\varphi \in T^*_t$. We distinguish two base cases, one where φ is a proposition, and another where φ is an atomic "do" formula.

(Base case 1) Suppose $\varphi = \chi_{t'}$ with $\chi \in \mathsf{Prop}$ and $t' \leq t$. $\Box_{t'}\chi_{t'}$ follows by applying Axiom A1. Then, apply Axiom A3 repeatedly until $\Box_t\chi_{t'}$ follows.

(Base case 2) Suppose $\varphi = do(a)_{t'}$ with $t' < t$ (note that $do(a)_t$ cannot occur in T^*_t because it does not occur in $Past(t)$). Using Axiom A4 and then repeatedly Axiom A3 we obtain $\Box_t do(a)_{t'}$.

(Conjunction) Suppose $\varphi = \psi \wedge \chi$. The induction hypothesis is $T^*_t \vdash \Box_t\psi$ and $T^*_t \vdash \Box_t\chi$, so therefore from $T^*_t \vdash \psi \wedge \chi$ we obtain $T^*_t \vdash \Box_t\psi \wedge \Box_t\chi$. Since $\Box_t\psi \wedge \Box_t\chi$ is equivalent to $\Box_t(\psi \wedge \chi)$, it follows directly that $T^*_t \vdash \Box_t(\psi \wedge \chi)$.

(Box) Suppose $\varphi = \Box_{t'}\psi$. By transitivity (which is not an axiom of our logic, but it holds in KT5): $\Box_{t'}\psi \rightarrow \Box_{t'}\Box_{t'}\psi$. Next, apply Axiom A3 repeatedly to obtain $\Box_t\Box_{t'}\varphi$.

(Negation) We make another case distinction on the negated formula. That is, we assume $T^*_t \vdash \neg\varphi$ and we show $T^*_t \vdash \Box_t\neg\varphi$, again by induction on the depth of the formula.

(Base case 1) Suppose $\varphi = \neg\chi_{t'}$ with $\chi \in \mathsf{Prop}$ and $t' \leq t$, then $\Box_t\neg\chi_t$ follows from Axiom A2 and A3 as described before.

(Base case 2) Suppose $\varphi = \neg do(a)_{t'}$ with $t' < t$. We apply Axiom A5 and A3 repeatedly until we have $\Box_t\neg do(a)_{t'}$.

(Conjunction) Suppose $\varphi = \neg(\psi \wedge \chi)$. The induction hypothesis is $T_t^* \vdash \Box_t \neg\psi$ and $T_t^* \vdash \Box_t \neg\chi$, which implies $T_t^* \vdash \Box_t \neg\psi \vee \Box_t \neg\chi$, which again implies $T_t^* \vdash \Box_t(\neg\psi \vee \neg\chi)$. Using De Morgan we obtain $T_t^* \vdash \Box_t \neg(\psi \wedge \chi)$.

(Box) Suppose $\varphi = \neg\Box_{t'}\psi$, which is equivalent to $\Diamond_{t'}\neg\psi$. From Axiom 5 we obtain $\Box_{t'}\Diamond_{t'}\neg\psi$, and by again applying Axiom A3 repeatedly we obtain $\Box_t \Diamond_{t'}\neg\psi$, which is equivalent to $\Box_t \neg\Box_{t'}\psi$, and this is what we had to show.

Lemma 5 (Valuation lemma). *For any maximal consistent set T^*, the following are true*

1. *T^* is deductively closed: $T^* \vdash \varphi$ implies that $\varphi \in T^*$;*
2. *$\varphi \in T^*$ iff $\neg\varphi \notin T^*$;*
3. *$\varphi \wedge \psi \in T^*$ iff $\varphi \in T^*$ and $\psi \in T^*$;*
4. *$\Box_t \varphi \in T^*$ iff for all $\overline{T}^*$ s.t. $T^* \equiv_t \overline{T}^*$: $\varphi \in \overline{T}^*$.*

Proof. 1. Because T^* is maximally consistent, either $\varphi \in T^*$ or $\neg\varphi \in T^*$. Suppose that $T^* \vdash \varphi$, and suppose for contradiction that $\neg\varphi \in T^*$. From this it follows that $T^* \vdash \neg\varphi$ and therefore $T^* \vdash \perp$, which would contradict consistency of T^*. Hence $\varphi \in T^*$.

2. Follows directly from the definition of a maximally consistent set.
3. Follows directly as well.
4. $\Rightarrow$: Suppose $\Box_t \varphi \in T^*$. Take arbitrary $\overline{T}^*$ with $T^* \equiv_t \overline{T}^*$. From Def. 2 (equivalent mcs) it follows that $\Box_t \varphi \in \overline{T}^*$. Therefore, by Axiom T we obtain $\varphi \in \overline{T}^*$.
$\Leftarrow$: We show this by contraposition. Therefore, suppose $\Box_t \varphi \notin T^*$. We will show that there exists some $\overline{T}^*$ with $T^* \equiv_t \overline{T}^*$ and $\varphi \notin \overline{T}^*$.
Suppose for contradiction that $\neg\varphi$ is not consistent with T_t^*, i.e. $T_t^* \cup \{\neg\varphi\} \vdash \perp$, so by the Deduction Theorem (Lemma 2), $T_t^* \vdash \varphi$ holds. By Lemma 3, we have (1) $\Box_t T_t^* \vdash \Box_t \varphi$. From Lemma 4 it follows that (2) $T_t^* \vdash \Box_t T_t^*$, so combining (1) and (2) gives $T_t^* \vdash \Box_t \varphi$. But this contradicts with our initial assumption that $\Box_t \varphi \notin T^*$. Thus, the assumption is invalid so $T_t^* \cup \{\neg\varphi\}$ is consistent.
By Lindenbaum's lemma, T_t^* can be extended to a mcs $\overline{T}^*$, and since $T_t^* \subseteq \overline{T}^*$, it follows directly that $\overline{T}^* \equiv_t T^*$. Therefore, there exists a mcs $\overline{T}^*$ with $\overline{T}^* \equiv_t T^*$ and $\varphi \notin \overline{T}^*$, and this is what we had to show.

We construct the canonical model by naming the states in our model as equivalence classes of mcs's, which are parameterized by a time point. For instance, the state $s = [T^*]_t$ is named as the set of mcs's equivalent to the mcs T^* up to and including time t. We then define accessibility relation between states named after equivalence classes up to and including subsequent time points of the same mcs. Finally, the valuation function assigns the set of propositions that are true in an equivalence class to the corresponding state.

Definition 4 (Canonical Tree). *Given a mcs T^*, we obtain a PAL-P-canonical tree $Tree_{T^*} = (S, R, v, act)$, where*

1. *$S = \bigcup_{t \in \mathbb{N}} S_t$ where $S_t = \{[\overline{T}^*]_t \mid \overline{T}^* \equiv_0 T^*\}$*
2. *sRs' iff $(\exists \overline{T}^*, t \in \mathbb{N}).(s = [\overline{T}^*]_t \wedge s' = [\overline{T}^*]_{t+1})$*
3. *$p \in v(s)$ iff $(\exists \overline{T}^*, t \in \mathbb{N}).(s = [\overline{T}^*]_t \wedge p_t \in \overline{T}^*)$.*
4. *$a = act((s, s'))$ iff $(\exists \overline{T}^*).(s = [\overline{T}^*]_t \wedge s' = [\overline{T}^*]_{t+1} \wedge do(a)_t \in \overline{T}^*)$*

Note that the existential quantifier in (3) of Def. 4 could equivalently be replaced by a universal quantifier, because of the definition of equivalence classes (Def. 3): All mcs's in $[\overline{T}^*]_t$ are equivalent up to time t, so if some timed proposition p_t is an element of some mcs in this set, then it is necessarily an element of any other mcs in this set as well.

Lemma 6. *Given a mcs T^*, $Tree_{T^*}$ is a tree.*

Proof. Suppose some T^* and let $Tree_{T^*} = (S, R, v, act)$. We have to show that R is serial, linearly ordered in the past, and connected.

- *serial*: Suppose some $s \in S$ s.t. $s = [T^*]t$. We have to show that there exists some $s' \in S$ such that sRs', i.e. there exists some $\overline{T'}^*$ s.t. $s = [\overline{T'}]_t$ and $s' = [\overline{T'}]_{t+1}$. This directly follows for $\overline{T'} = \overline{T}$ and by the fact that T^* is a mcs.
- *linearly ordered in the past:* Suppose some $s \in S$ s.t. $s = [T^*]t$ and $t > 0$ ($t = 0$ is the root of the tree). We show that there exists exactly one s' s.t. $s'Rs$. Suppose for contradiction that there exist $s', s'' \in S$ with $s' = [\overline{T'}]_{t-1}$ and $s'' = [\overline{T''}]_{t-1}$ such that $s' \neq s''$, i.e. $\overline{T'} \not\equiv_{t-1} \overline{T''}$. However, then $\overline{T'} \not\equiv_t \overline{T''}$ holds as well, but this contradicts with $s'Rs$ and $s''Rs$. Thus, $s' \neq s''$ is not possible.
- *connected:* Suppose $s, s' \in S$ with $s = [\overline{T}^*]_t$ and $s' = [\overline{T'}^*]_{t'}$. We show that there exists some s'' such that $s''R^*s$ and $s''R^*s'$, where R^* is the transitive closure of R. This directly holds for $s'' = [T^*]_0$, since then $s'' \in \{[\overline{T}^*]_0 \mid \overline{T}^* \equiv_0 T^*\}$.

Given a mcs T^*, we construct a path $\pi_{T^*} = (s_0, s_1, \ldots)$ from it by letting $s_t = [T^*]_t$. So $p \in v_p(\pi_{T^*t})$ iff $p_t \in T^*$ and $a = act(\pi_{T^*t})$ iff $do(a)_t \in T^*$.
Given a path π in a canonical tree $Tree_{T^*}$ and a $t \in \mathbb{N}$, we denote

$$T_{\pi|t} := \overline{T}^* \cap Past(t),$$

where $\overline{T}^* \in \pi_t$. Note that the definition is correct: $T_{\pi|t}$ does not depend on the choice of the element from T_π by Definition 2 and 3. We construct the set T_π from a path π as follows:

$$T_\pi = \bigcup_{t \in \mathbb{N}} T_{\pi|t}.$$

The next two lemmas show that for each π in the canonical tree $Tree_{T^*}$, T_π is a mcs.

Lemma 7. *Given a mcs T^*, For any path π in the canonical tree $Tree_{T^*}$: $T_{\pi|t} \subseteq T_{\pi|t+1}$.*

Proof. Suppose some mcs T^* and somoe arbitrary path π in the canonical tree $Tree_{T^*}$. From the construction of the canonical tree we have that $\pi_t R \pi_{t+1}$ iff there is some $\overline{T}^*$ with $\pi_t = [\overline{T}^*]_t$ and $\pi_{t+1} = [\overline{T}^*]_{t+1}$. Clearly, we have that $\overline{T}^*_t \subseteq \overline{T}^*_{t+1}$, and since $T_{\pi|t} \in [\overline{T}^*]_t$ and $T_{\pi|t} \in [\overline{T}^*]_{t+1}$, we also have that $T_{\pi|t} \subseteq T_{\pi|t+1}$.

Lemma 8. *Given a mcs T^*, for any path π in the canonical tree $Tree_{T^*}$: T_π is a mcs.*

Proof. (Consistent) From Lemma 7 we have that $T_{\pi|0} \subseteq \ldots \subseteq T_{\pi|t}$. Moreover, $T_{\pi|t} \subseteq \overline{T}^*$ where $\overline{T}^*$ is a mcs, which is consistent by definition, so T_π is consistent as well.
(Maximal) Suppose an arbitrary PAL-P-formula φ. Then there is a maximal t that appears in φ, therefore, $\varphi \in Past(t)$. By definition, $\varphi \in T_{\pi|t}$ or $\neg\varphi \in T_{\pi|t}$. Since $T_{\pi|t} \subseteq T_\pi$, we have that $\varphi \in T_\pi$ or $\neg\varphi \in T_\pi$. Hence T_π is maximal.

The following three lemmas are a direct consequence of the construction of T_π and π_T.

Lemma 9. *Given a mcs T^*, two paths π and π' in the canonical tree $Tree_{T^*}$ and some time point t: $\pi \sim_t \pi'$ iff. $T_\pi \equiv_t T_{\pi'}$.*

Lemma 10. *Given two mcs T^* and $\overline{T}^*$ and some time point t, $T^* \equiv_t \overline{T}^*$ iff. $\pi_{T^*} \sim_t \pi_{\overline{T}^*}$.*

Lemma 11. *Given a mcs T^*, in the canonical tree $Tree_{T^*}$,*
1. *For each π, $\pi_{(T_\pi)} = \pi$.*
2. *For each $\overline{T}^*$, $T_{(\pi_{\overline{T}^*})} = \overline{T}^*$.*

Note that by the previous lemma, for every path π in the canonical tree $Tree_{T^*}$, there exists a unique mcs $\overline{T^*}$ such that $\pi = \pi_{\overline{T^*}}$.

Lemma 12. *Given a mcs T^*: $(Tree_{T^*}, \pi_{\overline{T}^*})$ is a model.*

Proof. Suppose some T^*. From Lemma 6 we have that $Tree_{T^*}$ is a tree. In order to show that $(Tree_{T^*}, \pi_{\overline{T}^*})$ is a model we prove the three conditions on a model. Recall that

$$\pi_{T^*} = (s_0, s_1, \ldots) \text{ where } s_t = [T^*]_t.$$

1. Suppose $act(s_t) = a$. By the truth definition we have that $do(a) \in s_t$, so by the construction of s_t: $do(a)_t \in T^*$. By Axiom A9, $post(a)_{t+1} \in \overline{T}^*$, so $post(a) \in s_{t+1}$
2. Suppose $pre(\ldots, a, b)_t \in s_t$, so similarly we have $pre(\ldots, a, b) \in s_t$, and hence $pre(\ldots, a, b)_t \in T^*$. By Axiom A11, $pre(\ldots, a)_t \in T^*$ and so $pre(\ldots, a) \in s_t$.
3. We can prove this condition in the same way.

Lemma 13 (Truth lemma). *Given a mcs T^*: for every maximally PAL-P-consistent set of formula $\overline{T^*}$ and for every formula φ :*

$$(Tree_{T^*}, \pi_{\overline{T}^*}) \models \varphi \text{ iff. } \varphi \in \overline{T}^*$$

Proof. By induction on the depth of the proof.
(Base case) Suppose $\varphi = \chi_t$, for some atomic proposition $\chi \in \mathscr{L}_{PAL}$. From the truth definition we have $Tree_{T^*}, \pi_{\overline{T}^*} \models \chi_t$ iff. $\chi \in v(\pi_{\overline{T}^*t})$. From the construction of $\pi_{\overline{T}^*t}$ it follows then directly that $\chi_t \in \overline{T}^* t$. Suppose $\varphi = do(a)_t$. From the truth definition we have $Tree_{T^*}, \pi_{\overline{T}^*} \models do(a)_t$ iff $act(\pi_{\overline{T}^*t}) = a$. Again, from the construction of $\pi_{\overline{T}^*}$ we obtain $do(a)_t \in \overline{T}^*$.
(Negation) Suppose $\varphi = \neg\psi$. From the valuation lemma we know that $\neg\psi \in \overline{T}^*$ iff $\psi \notin \overline{T}^*$. By the induction hypothesis, $\psi \notin \overline{T}^*$ is equivalent to $Tree_{T^*}, \pi_{\overline{T}^*} \not\models \psi$. According to the truth definition that is equivalent to $Tree_{T^*}, \pi_{\overline{T}^*} \models \neg\psi$. Hence, $\neg\psi \in \overline{T}^*$ is equivalent to $Tree_{T^*}, \pi_{\overline{T}^*} \models \neg\psi$.
(Conjunction) Suppose $\varphi = \psi \wedge \chi$. From the valuation lemma we know that $\psi \wedge \chi \in \overline{T}^*$ iff $\psi \in \overline{T}^*$ and $\chi \in \overline{T}^*$. By the induction hypothesis, that is equivalent to $Tree_{T^*}, \pi_{\overline{T}^*} \models \psi$ and $Tree_{T^*}, \pi_{\overline{T}^*} \models \chi$, respectively. Lastly, applying the truth definition, this is equivalent to $Tree_{T^*}, \pi_{\overline{T}^*} \models \psi \wedge \chi$. Therefore, $\psi \wedge \chi \in \overline{T}^*$ iff $Tree_{T^*}, \pi_{\overline{T}^*} \models \psi \wedge \chi$.
(Necessity) Suppose $\varphi = \Box_t \psi$. We show both directions of the bi-implication separately.
$\Rightarrow$: Suppose that $Tree_{T^*}, \pi_{\overline{T}^*} \models \Box_t \psi$, i.e. for all π' with $\pi_{\overline{T}^*} \sim_t \pi' : Tree_{T^*}, \pi' \models \psi$. Pick such π' arbitrarily. From Lemma 11 we have that there is a unique mcs $\overline{\overline{T}}^*$ such that $\pi' = \pi_{\overline{\overline{T}}^*}$. Thus, $Tree_{T^*}, \pi_{\overline{\overline{T}}^*} \models \psi$ holds, and by the induction hypothesis, $\psi \in \pi(\overline{\overline{T}}^*)$ holds as well. Since $\pi(\overline{T}^*) \sim_t \pi(\overline{\overline{T}}^*)$, from Lemma 10 we obtain $\overline{T}^* \equiv_t \overline{\overline{T}}^*$. Thus, by the valuation lemma we obtain $\Box_t \psi \in \overline{T}^*$.
$\Leftarrow$: Suppose that $\Box_t \psi \in \overline{T}^*$. By the valuation lemma, for all $\overline{\overline{T}}^*$ with $\overline{T}^* \equiv_t \overline{\overline{T}}^* : \psi \in \overline{\overline{T}}^*$. Take such $\overline{\overline{T}}^*$ arbitrarily. From the induction hypothesis we have that $Tree_{T^*}, \pi_{\overline{\overline{T}}^*} \models \psi$. Since $\overline{T}^* \equiv_t \overline{\overline{T}}^*$, it follows from Lemma 10 that $\pi_{\overline{T}^*} \sim^t \pi_{\overline{\overline{T}}^*}$. Since $\overline{\overline{T}}^*$ was chosen arbitrarily, we have that for all π' with $\pi_{\overline{T}^*} \sim_t \pi'$ it holds that $Tree_{\overline{T}}, \pi' \models \psi$. Therefore, $Tree_{T^*}, \pi_{\overline{T}^*} \models \Box_t \psi$.

We can now prove that the logic PAL-P is strongly complete:

Proof (Theorem 1, Completeness). We prove this by contraposition, showing that $T \not\vdash \varphi$ implies $T \not\models \varphi$. If $T \not\vdash \varphi$, then $T \cup \{\varphi\}$ is inconsistent, so there is a mcs $T^* \supset T$ containing $\neg\varphi$, as the Lindenbaum lemma shows. By the Truth lemma we have that $Tree_{T^*}, \pi_{\overline{T}^*} \models \neg\varphi$ iff. $\neg\varphi \in \overline{T}^*$. And thus $Tree_{T^*}, \pi_{\overline{T}^*} \models \neg\varphi$, since $\neg\varphi \in \overline{T}^*$. Hence there is a model, namely $Tree$, and a path, namely $\pi_{\overline{T}^*}$, where $T \cup \{\neg\varphi\}$ is true and $T \cup \{\varphi\}$ is false. Therefore, $T \models \neg\varphi$, and that is what we had to show.

2 Coherence Condition Proofs

Lemma 1. *if $I' \subseteq I$, then $Cohere(I) \vdash Cohere(I')$.*

Proof. Suppose an intention database $I = \{(b_{t_1}, t_1), \ldots, (b_{t_n}, t_n)\}$ with $t_1 < \ldots < t_n$. Recall that

$$Cohere(I) = \Diamond_0 \bigvee_{\substack{a_k \in Act: k \notin \{t_1,\ldots,t_n\} \\ a_k = b_k: k \in \{t_1,\ldots,t_n\}}} pre(a_{t_1}, a_{t_1+1}, \ldots, a_{t_n})_{t_1}. \quad (1)$$

The case where $I' = I$ follows directly. Suppose $I' \subset I$. We consider three cases and show that for each case $Cohere(I) \vdash Cohere(I')$. We write $preform(I)$ to denote the precondition disjunction in the coherence condition, i.e. $Cohere(I) = \Diamond_0 preform(I)$. Moreover, if the subscript notation is omitted from a big disjunction, then it is equal to that of Eq. (1).

1. *Case 1: $I' = I \backslash \{(b_{t_1}, t_1)\}$*. Thus,

$$Cohere(I') = \Diamond_0 \bigvee pre(a_{t_2}, a_{t_2+1}, \ldots, a_{t_n})_{t_2}.$$

   Using Axiom (A11), we can derive $pre(a_{t_1})_{t_1}$ for each disjunct in $preform(I)$. Thus, we have that $preform(I)$ infers $pre(a_{t_1})_{t_1}$. From $pre(a_{t_1})_{t_1}$ using Axiom (A8) we obtain $\Diamond_{t_1} do(a_{t_1})_{t_1}$.
   Using (A1) and $preform(I)$, we derive

$$\bigvee_{\substack{a_k \in Act: k \notin \{t_1,\ldots,t_n\} \\ a_k = b_k: k \in \{t_1,\ldots,t_n\}}} \Box_{t_1} pre(a_{t_1}, \ldots, a_{t_n})_{t_1}. \quad (2)$$

   Note that $(\Box_{t_1}\varphi \vee \Box_{t_1}\psi) \rightarrow \Box_{t_1}(\varphi \vee \psi)$ is a theorem in our logic. Combining this with Eq. (2) gives $\Box_{t_1} preform(I)$. The K-axiom is equivalent to $(\Box_{t_1}\varphi \wedge \Diamond_{t_1}\psi) \rightarrow \Diamond_{t_1}(\varphi \wedge \psi)$. Therefore, from $\Diamond_{t_1} do(a_{t_1})_{t_1}$ and $\Box_{t_1} preform(I)$ we obtain $\Diamond_{t_1}(do(a_{t_1})_{t_1} \wedge preform(I))$. Let $\varphi = do(a_{t_1})_{t_1}$ and $preform(I) = \psi_1 \vee \ldots \vee \psi_n$. We can rewrite $\Diamond_{t_1}(\varphi \wedge (\psi_1 \vee \ldots \vee \psi_n)$ to $\Diamond_{t_1}(\varphi \wedge \psi_1) \vee \ldots \vee \Diamond_t(\varphi \wedge \psi_n)$. Using Axiom (A12), we obtain

$$\Diamond_{t+1} \bigvee pre(a_{t_1+1}, \ldots, a_{t_n})_{t_1+1} \quad (3)$$

   In each of the disjuncts of Eq. (3). In case $t_1 + 1 < t_2$, we make another case distinction on a_{t_1+1}: for each case we apply the same procedure as before and we obtain $pre(a_{t_1+2}, \ldots, a_{t_n})_{t_1+2}$ in each case distinction. Thus we can derive this formula. We can repeat this procedure until $t_1 + i = t_2$. This means we have shown that

$$preform(I) \rightarrow \Diamond_{t_2} preform(I'). \quad (4)$$

   We remains to show is that $\Diamond_0 preform(I) \rightarrow \Diamond_0 preform(I')$. Let $\varphi = preform(I)$ and $\psi = \Diamond_{t_2} preform(I')$. Using Necessitation and contraposition, we have $\Box_0(\neg\psi \rightarrow \neg\varphi)$. Using the K-axiom and contraposition again, we obtain $\Diamond_0\varphi \rightarrow \Diamond_0\psi$, i.e. $Cohere(I) \rightarrow \Diamond_0\Diamond_{t_2} preform(I')$. $\Diamond_0\Diamond_{t_2} preform(I')$ derives $\Diamond_0 preform(I')$, so $Cohere(I) \rightarrow Cohere(I')$ is a theorem. Hence, by the deduction theorem, $Cohere(I) \vdash Cohere(I')$.
2. *Case 2: $I' = I \backslash \{(b_{t_n}, t_n)\}$*. Using Axiom (A11).
3. *Case 3: $I' = I \backslash \{(b_{t_i}, t_i)\}$* with $t_1 < i < t_n$. Note that $preform(I) \rightarrow preform(I')$ is a theorem of our logic. Thus, using the same technique as in case 1, we obtain $\vdash Cohere(I) \rightarrow Cohere(I')$, and again by the deduction theorem, $Cohere(I) \vdash Cohere(I')$.

Proposition 1. *If an agent $A = (B, I)$ is coherent, then $WB(B, I)$ is consistent.*

Proof. First we show that $pre(a_0, \ldots, a_m)_t \vdash \Diamond_t(do(a)_t \wedge do(a_1)_{t+1} \wedge \ldots \wedge do(a_m)_{t+m})$. Therefore, suppose $(1) pre(a_0, \ldots, a_m)_t$. Applying Axiom A11 m times gives $pre(a_0)_t$. By Axiom A8 we obtain (2) $\Diamond_t do(a_0)_t$. From (1) and (2) and by the fact that $pre(a_0, \ldots, a_m)_t \equiv \Diamond_t pre(a_0, \ldots, a_m)_t$, we derive (3) $\Diamond_t(pre(a_{t+1}, \ldots, a_m)_{t+1} \wedge do(a_0)_t)$. Applying the same procedure to $pre(a_{t+1}, \ldots, a_m)_{t+1}$ iteratively until there are not *pre* formulas left, we obtain $\Diamond_t(do(a_0)_t \wedge \Diamond_{t+1}(do(a_1)_{t+1} \wedge \ldots))$. By taking the contrapositive of Axiom A3 repeatedly we obtain $\Diamond_t(do(a_0)_t \wedge \Diamond_t(do(a_1)_{t+1} \wedge \ldots))$. Rewriting this and using transitivity (i.e. $\Diamond_t\Diamond_t\varphi \rightarrow \Diamond_t\varphi$, which isn't an axiom but can be derived from KT5) we obtain $\Diamond_t(do(a)_t \wedge do(a_1)_{t+1} \wedge \ldots \wedge do(a_m)_{t+m})$, i.e. $\Diamond_t \bigwedge_{k=0}^m do(a_k)_{t+k}$.
Next, let $I = \{(b_{t_1}, t_1), \ldots, (b_{t_n}, t_n)\}$ with $t_1 < \ldots < t_n$. By the fact that $pre(a_0, \ldots, a_m)_t \vdash \bigwedge_{k=0}^m do(a_k)_{t+k}$, $Cohere(I)$ (Def. 11) implies

$$\Diamond_0 \bigvee_{\substack{a_k \in Act: k \notin \{t_1,\ldots,t_k\} \\ a_k = b_k: k \in \{t_1,\ldots,t_n\}}} \Diamond_{t_1}(do(a_{t_1})_{t_1} \wedge do(a_{t_1+1})_{t_1+1} \wedge \ldots \wedge do(a_{t_n})_{t_n}$$

Consequently, $Cohere(I)$ implies $\Diamond_0\Diamond_{t_1} \bigwedge_{k=1}^n do(b_{t_k})_{t_k}$, and by (A3) and transitivity this implies $\Diamond_0 \bigwedge_{(a,t)\in I} do(a)_t$. Therefore, if (B, I) is coherent, then the set $B \cup \{\Diamond_0 \bigwedge_{(a,t)\in I} do(a)_t\}$ is consistent. By the completeness theorem, this means that there exists a model $m = (T, \pi)$ such that $m \models B$ and $m \models \Diamond_0 \bigwedge_{(a,t)\in I} do(a)_t$. Since B is a strong belief set, it follows that for all $\pi' \in T : T, \pi' \models B$. Since $m \models \Diamond_0 \bigwedge_{(a,t)\in I} do(a)_t$, there exists some $\pi'' \in T$ with $T, \pi'' \models \bigwedge_{(a,t)\in I} do(a)_t$. But then also $T, \pi'' \models B$, so there exists some model (T, π'') with $T, \pi'' \models B$ and $T, \pi'' \models \bigwedge_{(a,t)\in I} do(a)_t$. By the completeness theorem we obtain that $B \cup \{\bigwedge_{(a,t)\in I} do(a)_t\}$ is consistent, hence $WB(B, I)$ is consistent.

Proposition 2 (Coherence Condition 2b). *If $\{(a,t), (b, t+1)\} \subseteq I$ and B is a set of strong beliefs such that (B, I) coherent, then $\{pre(b)_{t+1}, post(a)_{t+1}\}$ is consistent with B.*

Proof. From Lemma 1 it follows that we can assume $I = \{(a,t),(b,t+1)\}$ without loss of generality. Let B be a set of strong beliefs whose set of models is M. We need to show that consistency of $\{pre(b)_{t+1}, post(a)_{t+1}\}$ with B follows from our coherence condition. Note that the coherence formula is $Cohere(I) = \Diamond_0 pre(a,b)_t$. By the axiom (A12) (Definition 4) and the Deduction theorem we have $pre(a,b)_t \wedge do(a)_t \vdash pre(b)_{t+1}$. Using the axiom (A9): $do(a)_t \rightarrow post(a)_{t+1}$ from PAL (Def. 7 [1]) and Deduction theorem we obtain $do(a)_t \vdash post(a)_{t+1}$. Consequently,

$$pre(a,b)_t \wedge do(a)_t \vdash pre(b)_{t+1} \wedge post(a)_{t+1}.$$

Thus, in order to prove that $\{pre(b)_{t+1}, post(a)_{t+1}\}$ is consistent with B, it is sufficient to show that $pre(a,b)_t \wedge do(a)_t$ is consistent with B. If (B,I) is coherent, then there is a model $m = (T,\pi) \in M$ such that $m \models Cohere(I)$, so there is $\pi' \in T$ such that $(T,\pi') \models pre(a,b)_t$. By Definition 3.2, then there exist $\pi'' \in T$ such that $\pi' \sim_t \pi''$ and $act(\pi''_t) = a$. Then $(T,\pi'') \models pre(a,b)_t \wedge do(a)_t$. Since M is a set of models of strong beliefs, we obtain $(T,\pi'') \in M$, i.e., (T,π'') is also a model of B. Then B is consistent with $pre(a,b)_t \wedge do(a)_t$, by Completeness theorem.

3 Representation Theorems Proofs

Recall that $Mod(\varphi)$ is the set of models of φ. Similarly, given some t-restricted PAL-P formula φ, we define $Mod^{|t}(\varphi)$ as the set of t-restricted models of φ. Recall from the paper that $Ext(M^{|t}_{SB})$ is the set of all possible extensions of a set of bounded model of strong beliefs $M^{|t}_{SB}$ to models, i.e.

$$Ext(M^{|t}_{SB}) = \{m \in \mathbb{M} \mid m^{|t} \in M^{|t}_{SB}\}.$$

Note that Ext is defined on the sets of bounded models of **strong beliefs** only. In order to simplify notation in the proof of the representation theorem, we define the following abbreviation:

Given a set of restricted modesls $\{m^{|t}_1,\ldots,m^{|t}_n\}$, with $m^{|t}_i = (T^{|t}_i,\pi^{|t}_i)$, we introduce $Ext(m^{|t}_1,\ldots,m^{|t}_n)$ as:

$$Ext(m^{|t}_1,\ldots,m^{|t}_n) := Ext(\{(T^{|t},\pi^{|t}) \mid \bigvee_{k=1}^{n} T^{|t} = T^{|t}_i\}). \tag{5}$$

Lemma 2. *Given a set of t-bounded models of strong beliefs $M^{|t}_{SB}$, there exists a strong belief formula $form(M^{|t}_{SB})$ such that $Mod(form(M^{|t}_{SB})) = Ext(M^{|t}_{SB})$.*

Proof. For a given $T^{|t}$ we define the strong belief formula

$$form(T^{|t}) = \bigwedge_{\pi^{|t\prime} \in T^{|t}} \Diamond_0 \alpha_{\pi^{|t\prime}} \wedge \bigwedge_{\pi^{|t\prime} \notin T^{|t}} \neg\Diamond_0 \alpha_{\pi^{|t\prime}},$$

where

$$\alpha_{\pi^{|t}} = \bigwedge_{n=0}^{t} \left(\bigwedge_{\chi \in v(\pi^{|t}{}_n)} \chi_n \wedge \bigwedge_{\chi \notin v(\pi^{|t}{}_n)} \neg\chi_n \wedge \bigwedge_{act(\pi^{|t}{}_n)=a} do(a)_n \right).$$

Intuitively, $form(T^{|t})$ is a strong belief formula describing all of the paths of T up to t. Each $\alpha_{\pi^{|t}}$ is a formula describing the path π up to t: It contains all propositions that are true and false at each time moment, and all actions that are executed. Note that Axiom A7 of PAL-P ensures that only one action can be executed per time moment.
Let T' be a tree. From the construction of the formula $form(T^{|t})$ it follows that if $T^{|t\prime} = T^{|t}$, then for every $\pi' \in T'$ we have $(T',\pi') \models form(T^{|t})$. On the other hand, if $T^{|t\prime} \neq T^{|t}$, then there is π such that either $\pi^{|t} \in T^{|t} \setminus T^{|t\prime}$ or $\pi^{|t} \in T^{|t\prime} \setminus T^{|t}$. Suppose that $\pi^{|t} \in T^{|t} \setminus T^{|t\prime}$. Then for any $\pi \in T'$ we have $(T',\pi') \not\models \Diamond_0 \alpha_{\pi^{|t}}$, so $(T',\pi') \not\models form(T^{|t})$. Similarly, if $\pi^{|t} \in T^{|t\prime} \setminus T^{|t}$, then $(T',\pi) \not\models \neg\Diamond_0 \alpha_{\pi^{|t}}$, so again we have $(T',\pi) \not\models form(T^{|t})$. Thus, we proved

$$Mod(form(T^{|t})) = Ext(\{(T^{|t},\pi^{|t}) \in \mathbb{M}^{|t} \mid \pi^{|t} \in T^{|t}\}).$$

Now we define

$$form(M^{|t}{}_{SB}) = \bigvee\{form(T^{|t}) \mid (T^{|t},\pi^{|t}) \in M^{|t}{}_{SB}\}.$$

Note that our set of propositional letters is finite, and that we have finitely many deterministic actions, so $M^{|t}{}_{SB}$ is a finite set. Consequently, the above disjunction is finite.
Finally, we have

$$\begin{aligned} & Mod(form(M^{|t}{}_{SB})) \\ = & \bigcup\{Mod(form(T^{|t})) \mid (T^{|t},\pi^{|t}) \in M^{|t}{}_{SB}\} \\ = & \bigcup Ext(\{(T^{|t},\pi^{|t}) \in M^{|t}{}_{SB} \mid \pi^{|t} \in T^{|t}\}) \\ = & Ext(M^{|t}{}_{SB}). \end{aligned}$$

□

Note that $form$ is defined on the set of bounded models of **strong beliefs** only. In order to simplify notation in the proof of the representation theorem, we define the following abbreviation:

Given a set of models $\{m_1,\ldots,m_n\}$, with $m_i=(T_i,\pi_i)$, we introduce $form(m_1,\ldots,m_n)$ as:

$$form(m_1,\ldots,m_n) := form(\{(T^{|t},\pi^{|t}) \mid \bigvee_i^n T^{|t} = T_i^{|t}\}). \tag{6}$$

Lemma 3. *If $\varphi \in B_t$ and $T_1^{|t} = T_2^{|t}$, then $(T_1,\pi_1) \models \varphi$ iff $(T_2,\pi_2) \models \varphi$.*

Proof. By induction on the complexity of φ.

Corollary 1. *Given a t-bounded strong belief set B, there exists a formula ψ such that $B = \{\varphi \mid \psi \vdash \varphi\}$.*

Proof. For a given belief set B, from Lemma 3 follows that $Mod^{|t}(B)$ is a set of t-bounded models of a strong beliefs such that $Ext(Mod^{|t}(B)) = Mod(B)$. If $\psi = form(Mod^{|t}(B))$, then form Lemma 2 we obtain $Mod(\psi) = Mod(B)$, and by the completeness theorem, $B = Cl(\psi)$. □

Note that for a formula $\varphi \in B_t$, the satisfiability of the formula in a model m depends only on the paths in its restricted counterpart $m^{|t}$, for a set of intentions bounded up to t so we can write that

$$(M^{|t},I) \text{ is coherent iff } (\mathrm{M},\mathrm{I}) \text{ is coherent.} \tag{7}$$

Definition 5 (Faithful assignment). *A* faithful assignment *is a function that assigns to each strong belief formula $\psi \in \mathbb{B}^{|t}$ a total pre-order $\leq^t_\psi$ over $\mathbb{M}$ and to each intention database $I \in \mathbb{D}^{|t}$ a selection function γ^t_I and satisfies the following conditions:*

1. *If $m_1,m_2 \in Mod(\psi)$, then $m_1 \leq^t_\psi m_2$ and $m_2 \leq^t_\psi m_1$.*
2. *If $m_1 \in Mod(\psi)$ and $m_2 \notin Mod(\psi)$, then $m_1 < m_2$.*
3. *If $\psi \equiv \phi$, then $\leq^t_\psi = \leq^t_\phi$.*
4. *If $T^{|t} = T_2^{|t}$, then $(T,\pi) \leq^t_\psi (T_2,\pi_2)$ and $(T_2,\pi_2) \leq^t_\psi (T,\pi)$.*

Theorem 2 (Representation Theorem). *An agent revision operator $*_t$ satisfies postulates (P1)-(P12) iff there exists a faithful assignment that maps each ψ to a total pre-order $\leq^t_\psi$ and each I to a selection function γ^t_I such that if $(\psi,I) *_t (\varphi,i) = (\psi',I')$, then:*

1. *$Mod(\psi') = \min(Mod(\varphi),\leq^t_\psi)$*
2. *$I' = \gamma^t_I(Mod(\psi'),i)$*

Proof. "$\Rightarrow$": Suppose that some agent revision operator $*_t$ satisfies postulates (P1)-(P12). Given models m_1 and m_2, let $(\psi,\emptyset) *_t (form(m_1,m_2),\varepsilon) = (\psi',\emptyset)$ (note that we use the abbreviation (6) for $form$). We define $\leq^t_\psi$ by $m_1 \leq^t_\psi m_2$ iff $m_1 \models \psi$ or $m_1 \models \psi'$. We also define γ^t_I by $\gamma^t_I(M^{|t}_{SB},i) = I'$, where $(form(M^{|t}_{SB}),I) *_t (\top,i) = (\psi_2,I')$ (note that $\psi_2 \equiv form(M^{|t}_{SB})$).
In order to prove that the assignment is faithful for $\leq^t_\psi$, we define $\psi \circ_t \varphi = \psi'$ when $(\psi,\emptyset) *_t (\varphi,\varepsilon) = (\psi',\emptyset)$. We can now prove that postulates (P1)-(P6) for $*_t$ imply the Katsuno and Mendelzon postulates (R1)-(R6).

(R1) $\psi \circ_t \varphi$ implies φ
(R2) If $\psi \wedge \varphi$ is satisfiable, then $\psi \circ_t \varphi \equiv \psi \wedge \varphi$
(R3) If φ is satisfiable, then $\psi \circ_t \varphi$ is also satisfiable
(R4) If $\psi \equiv \psi'$ and $\varphi \equiv \varphi'$, then $\psi \circ_t \varphi \equiv \psi' \circ_t \varphi'$
(R5) $(\psi \circ_t \varphi) \wedge \varphi'$ implies $\psi \circ_t (\varphi \wedge \varphi')$
(R6) If $(\psi \circ_t \varphi) \wedge \varphi'$ is satisfiable, then $\psi \circ_t (\varphi \wedge \varphi')$ implies $(\psi \circ_t \varphi) \wedge \varphi'$

We show that (R5) holds, i.e. $(\psi \circ_t \varphi) \wedge \varphi'$ implies $\psi \circ_t (\varphi \wedge \varphi')$ holds, the other cases are similar. Let us denote with ψ' the formula $\psi \circ_t \varphi$, with $\overline{\psi}'$ the formula $\psi \circ_t (\varphi \wedge \varphi')$, and with $\overline{\varphi}$ the formula $\varphi \wedge \varphi'$. Then we have $(\psi,\emptyset) *_t (\varphi,\varepsilon) = (\psi',\emptyset)$ and $(\psi,\emptyset) *_t (\overline{\varphi},\varepsilon) = (\overline{\psi}',\emptyset)$. Then by postulate (P5), $\psi' \wedge \varphi'$ implies $\overline{\psi}'$, or equivalently $(\psi \circ_t \varphi) \wedge \varphi'$ implies $\psi \circ_t (\varphi \wedge \varphi')$.
Modifying the proof technique of Katsuno and Mendelzon, we show that 1) $\leq^t_\psi$ is a total pre-order, 2) the assignment ψ to $\leq^t_\psi$ is faithful, 3) $Mod(\psi') = \min(Mod(\varphi),\leq^t_\psi)$. Then we show that 4) γ^t_I is a selection function and 5) $I' = \gamma^t_I(Mod^{|t}(\psi'),i)$.

1. To show: $\leq^t_\psi$ is a total pre-order.
   - To show: Totality and reflexivity. From (R1) and (R3): $Mod(\psi \circ_t form(m_1,m_2))$ is a nonempty subset of $Ext(m_1^{|t},m_2^{|t})$ (note that we use the abbreviation (5) for Ext). Therefore, for each $m \in Ext(m_1^{|t})$ and $m' \in Ext(m_2^{|t})$, we have that either $m \leq^t_\psi m'$ or $m' \leq^t_\psi m$. We now show, without loss of generality, that for each $m,m' \in Ext(m_1^{|t})$, both $m \leq^t_\psi m'$ and $m' \leq^t_\psi m$ hold. Therefore, let $m,m' \in Ext(m_1^{|t})$, so $m^{|t} = m^{|t\prime}$. By Lemma 3 of the paper, $Mod(form(m^{|t})) = Ext(m^{|t}) = Ext(m^{|t\prime}) = Mod(form(m^{|t\prime}))$. Hence, $form(m) \equiv form(m')$, so $form(m,m') \equiv form(m)$. By (R4): $Mod(\psi \circ_t form(m^{|t})) = Mod(\psi \circ_t form(m,m')))$. By (R1), $m \in Mod(\psi \circ_t form(m)$, so $m \in Mod(\psi \circ_t form(m,m'))$. Hence, by the definition of $\leq^t_\psi$: $m \leq^t_\psi m'$. We can prove $m' \leq^t_\psi m$ similarly. This proves that $\leq^t_\psi$ is total, which implies reflexivity.
   - To show: Transitivity. Assume $m_1 \leq^t_\psi m_2$ and $m_2 \leq^t_\psi m_3$. We show $m_1 \leq^t_\psi m_3$. There are three cases to consider:

(a) $m_1 \in Mod(\psi)$. $m_1 \leq^t_\psi m_3$ follows from the definition of $\leq^t_\psi$.

(b) $m_1 \notin Mod(\psi)$ and $m_2 \in Mod(\psi)$. Since $Mod(\psi \wedge form(m_1,m_2)) = Ext(m_2^{|t})$ holds, $Mod(\psi \circ_t form(m_1,m_2)) = Ext(m_2^{|t})$ follows from (R2). Thus $m_1 \not\leq^t_\psi m_2$ follows from $m_1 \notin Mod(\psi)$. This contradicts $m_1 \leq^t_\psi m_2$, so this case is not possible.

(c) $m_1 \notin Mod(\psi)$ and $m_2 \notin Mod(\psi)$. By (R1) and (R3), $Mod(\psi \circ_t form(m_1,m_2,m_3))$ is a nonempty subset of $Ext(m_1^{|t}, m_2^{|t}, m_3^{|t})$. We now consider two subcases.

i. $Mod(\psi \circ_t form(m_1,m_2,m_3)) \cap Ext(m_1^{|t}, m_2^{|t}) = \emptyset$. In this case, $Mod(\psi \circ_t form(m_1,m_2,m_3)) = Ext(m_3^{|t})$ holds. If we regard φ and φ' as $form(m_1,m_2,m_3)$ and $form(m_2,m_3)$ respectively in Conditions (R5) and (R6), we obtain

$$Mod(\psi \circ_t form(m_1,m_2,m_3)) \cap Ext(m_2^{|t}, m_3^{|t}) = Mod(\psi \circ_t form(m_2,m_3)).$$

Hence, $Mod(\psi \circ_t form(m_2,m_3)) = Ext(m_3^{|t})$. This contradicts $m_2 \leq^t_\psi m_3$ and $m_2 \notin Mod(\psi)$. Thus, this subcase is not possible.

ii. $Mod(\psi \circ_t form(m_1,m_2,m_3)) \cap Ext(m_1^{|t}, m_2^{|t}) \neq \emptyset$. Since $m_1 \leq^t_\psi m_2$ and $m_1 \notin Mod(\psi)$, $m_1 \in Mod(\psi \circ_t form(m_1,m_2))$ holds. Hence, by regarding φ and φ' as $form(m_1,m_2,m_3)$ and $form(m_1,m_2)$ respectively in Conditions (R5) and (R6), we obtain

$$Mod(\psi \circ_t form(m_1,m_2,m_3)) \cap Ext(m_1^{|t}, m_2^{|t}) = Mod(\psi \circ_t form(m_1,m_2)).$$

Thus,

$$m_1 \in Mod(\psi \circ_t form(m_1,m_2,m_3)) \cap Ext(m_1^{|t}, m_2^{|t})$$

holds. By using conditions (R5) and (R6) again in a similar way, we can obtain $m_1 \in Mod(\psi \circ_t form(m_1,m_3))$. Therefore, $m_1 \leq^t_\psi m_3$ holds.

2. To show: The assignment mapping ψ to $\leq^t_\psi$ is faithful. We prove the four conditions separately
   (a) The first condition follows from the definition of $\leq^t_\psi$.
   (b) For the second condition, assume that $m \in Mod(\psi)$ and $m' \notin Mod(\psi)$. Then $Mod(\psi \circ_t form(m,m')) = Ext(m^{|t})$ follows from (R2). Therefore, $m <^t_\psi m'$ holds.
   (c) The third condition follows from (R4).
   (d) For the fourth condition, for $m_1 = (T_1, \pi_1)$ and $m_2 = (T_2, \pi_2)$ such that $T_1^{|t} = T_2^{|t}$, let ψ' be as above. Since $\psi, \psi' \in B_t$, by Lemma 3 we obtain $m_1 \models \psi$ iff $m_2 \models \psi$ and $m_1 \models \psi'$ iff $m_2 \models \psi'$, so $m_1 \leq^t_\psi m_2$ and $m_2 \leq^t_\psi m_1$.
3. To show: $Mod(\psi') = \min(Mod(\varphi), \leq^t_\psi)$. Note that this can be equivalently rewritten as $Mod(\psi \circ_t \varphi) = \min(Mod(\varphi), \leq^t_\psi)$. If φ is unsatisfiable then both are empty. So we assume φ is satisfiable. We show both containments separately.
   – To show: $Mod(\psi \circ_t \varphi) \subseteq \min(Mod(\varphi), \leq^t_\psi)$. Assume for contradiction that $m \in Mod(\psi \circ_t \varphi)$ and $m \notin \min(Mod(\varphi), \leq^t_\psi)$. By condition (R1), m is a model of φ. Hence, there is a model m' of φ such that $m' <^t_\psi m$. We consider two cases:
     (a) $m' \in Mod(\psi)$. Since $m' \in Mod(\varphi)$, $\psi \wedge \varphi$ is satisfiable. Hence, by condition (R2), $\psi \circ_t \varphi \equiv \psi \wedge \varphi$ holds. Thus, $m \in Mod(\psi)$ follows from $m \in Mod(\psi \circ_t \varphi)$. Therefore, $m \leq^t_\psi m'$ holds. This contradicts $m' <^t_\psi m$.
     (b) $Mod(\psi \circ_t form(m,m')) = Ext(m^{|t})$. Since both m and m' are models of φ, $\varphi \wedge form(m,m') \equiv form(m,m')$ holds. Thus,

$$Mod(\psi \circ_t \varphi) \cap Ext(m^{|t}, m^{|t\prime}) \subseteq Mod(\psi \circ_t form(m,m'))$$

     follows from condition (R5). Since we assume $Mod(\psi \circ_t form(m,m')) = Ext(m^{|t\prime})$, we obtain $m \notin Mod(\psi \circ_t \varphi)$. This is a contradiction.
   – To prove: $\min(Mod(\varphi), \leq^t_\psi) \subseteq Mod(\psi \circ_t \varphi)$. Assume for contradiction that $m \in \min(Mod(\varphi), \leq^t_\psi)$ and $m \notin Mod(\psi \circ_t \varphi)$. Since we also assume that φ is satisfiable, it follows from condition (R3) that there is an interpretation m' such that $m' \in Mod(\psi \circ_t \varphi)$. Since both m and m' are models of φ, $form(m,m') \wedge \varphi \equiv form(m,m')$ holds. By using conditions (R5) and (R6), we obtain

$$Mod(\psi \circ_t \varphi) \cap Ext(m^{|t}, m^{|t\prime}) = Mod(\psi \circ_t form(m,m')).$$

   Since $m \notin Mod(\psi \circ_t \varphi)$, $Mod(\psi \circ_t form(m,m')) = Ext(m^{|t\prime})$ holds. Hence, $m' \leq^t_\psi m$ holds. On the other hand, since m is minimal in $Mod(\varphi)$ with respect to $\leq^t_\psi$, $m \leq^t_\psi m'$ holds. Since $Mod(\psi \circ_t form(m,m')) = Ext(m^{|t\prime})$, $m \in Mod(\psi)$ holds. Therefore, $m \in Mod(\psi \circ_t \varphi)$ follows from condition (R2). This is a contradiction.
4. To show: γ^t_I is a selection function. This is direct consequence of the completeness theorem and the postulates (P7)-(P10) and (P12), taking into account (3). For example, if (P7) holds, then ψ' is consistent with $Cohere(I')$, so by completeness there is a model of both ψ' and $Cohere(I')$. since $Mod(\psi') = \min(Mod(\varphi), \leq^t_\psi)$, we obtain that $(Mod(\psi'), I')$ is coherent.
5. To show: $I' = \gamma^t_I(Mod^{|t}(\psi'), i)$. By our definition of γ^t_I we have that $(\psi', I) *_t (\top, i) = (\overline{\psi}, \gamma^t_I(Mod(\psi'), i))$ (recall that $\psi' \equiv \psi_2$). Since $(\psi, I) *_t (\varphi, i) = (\psi', I')$, by (P11) we obtain that $I' = \gamma^t_I(Mod(\psi'), i)$.

"$\Leftarrow$": Assume that there is a faithful assignment that maps ψ to a total pre-order $\leq^t_\psi$ and I to a selection function γ^t_I. We define the t-bounded revision operator $*_t$ as follows:

$$(\psi, C) *_t (\overline{\varphi}, c) = (form(\min(Mod^{|t}(\varphi), \leq^t_\psi), \gamma^t_I(\min(Mod^{|t}(\varphi), \leq^t_\psi), i)).$$

First we show that the operator is well correctly defined, i.e. that $\min(Mod^{|t}(\varphi), \leq^t_\psi)$ is a set of t-bounded models of strong beliefs. Let $\overline{T} = \overline{T}'$. Since $\varphi \in \mathbb{B}^{|t}$, by Lemma 3 we obtain that $(T,\pi) \models \varphi$ iff $(T',\pi') \models \varphi$. Now suppose that $(T,\pi) \in \min(Mod(\varphi), \leq^t_\psi)$. If $(T',\pi') \notin \min(Mod(\varphi), \leq^t_\psi)$, then $(T,\pi) < (T',\pi')$, which is impossible by the definition of faithful assignment. Thus, $\min(Mod^{|t}(\varphi), \leq^t_\psi)$ is a set of t-bounded models of strong beliefs.
Let us now prove that $*_t$ satisfies the postulates (P1)-(P-12). In order to prove the first six postulates, we define the operator $\circ_t$ by $\psi \circ_t \varphi = form(\min(Mod^{|t}(\varphi), \leq^t_\psi))$. Let use show that $\circ_t$ satisfies conditions (R1)-(R6) of KM (see the ($\Rightarrow$) part of the proof). It is obvious that condition (R1) follows from the definition of the revision operator $\circ_t$. It is also obvious that conditions (R3) and (R4) follow from the definition of the faithful assignment. What remains to show is condition (R2), (R5), and (R6).

- To prove: (R2). It suffices to show if $Mod(\psi \wedge \varphi)$ is not empty then $Mod(\psi \wedge \varphi) = \min(Mod(\varphi), \leq^t_\psi)$. $Mod(\psi \wedge \varphi) \subseteq \min(Mod(\varphi), \leq^t_\psi)$ follows from the conditions of the faithful assignment. To prove the other containment, we assume that $m \in \min(Mod(\varphi), \leq^t_\psi)$ and $m \notin Mod(\varphi \wedge \varphi)$. Since $Mod(\psi \wedge \varphi)$ is not empty, there is a model $m' \in Mod(\psi \wedge \varphi)$. Then $m \not\leq^t_\psi m'$ follows from the conditions of the faithful assignment. Moreover, $m' \leq^t_\psi m$ follows from the conditions of the faithful assignment. Hence, m is not minimal in $Mod(\varphi)$ with respect to $\leq^t_\psi$. This is a contradiction.
- To prove: (R5) and (R6). It is obvious that if $(\psi \circ_t \varphi) \wedge \varphi'$ is unsatisfiable then (R6) holds. Hence, it suffices to show that if $\min(Mod(\mu), \leq^t_\psi) \cap Mod(\varphi')$ is not empty then

$$\min(Mod(\varphi), \leq^t_\psi) \cap Mod(\varphi') = \min(Mod(\varphi \wedge \varphi'), \leq^t_\psi)$$

holds. Assume that $m \in \min(Mod(\varphi), \leq^t_\psi) \cap Mod(\varphi')$ and $m \notin \min(Mod(\varphi \wedge \varphi'), \leq^t_\psi)$. Then, since $m \in Mod(\varphi \wedge \varphi')$, there is an interpretation m' such that $m' \in Mod(\varphi \wedge \varphi')$ and $m' <_\psi m$. This contradicts $m \in \min(Mod(\varphi), \leq^t_\psi)$. Therefore, we obtain

$$\min(Mod(\varphi), \leq^t_\psi) \cap Mod(\varphi') \subseteq \min(Mod(\varphi \wedge \varphi'), \leq^t_\psi).$$

To prove the other containment, we assume that $m \notin \min(Mod(\varphi), \leq^t_\psi) \cap Mod(\varphi')$ and $m \in \min(Mod(\varphi \wedge \varphi'), \leq^t_\psi)$. Since $m \in Mod(\varphi'), m \notin \min(Mod(\varphi), \leq^t_\psi)$ holds. Since we assume that $\min(Mod(\varphi), \leq^t_\psi) \cap Mod(\varphi')$ is not empty, suppose that m' is an element of $\min(Mod(\varphi), \leq^t_\psi) \cap Mod(\varphi')$. Then $m' \in Mod(\varphi \wedge \varphi')$ holds. Since we assume that $m \in \min(Mod(\varphi \wedge \varphi'), \leq^t_\psi)$ and $\leq^t_\psi$ is total, $m \leq^t_\psi m'$ holds. Thus, $m \in \min(Mod(\varphi), \leq^t_\psi)$ follows from $m' \in \min(Mod(\varphi), \leq^t_\psi)$. This is a contradiction.

Note that the conditions (R1)-(R6) imply the conditions (P1)-(P6). For example, suppose (R3) and let $(\psi, I) *_t (\varphi, i) = (\psi', I')$. Then $\psi' = form(\min(Mod(\varphi), \leq^t_\psi) \cap Mod(\varphi')) = \psi \circ_t \varphi$ so if φ is satisfiable, ψ' is satisfiable as well. Thus, (P3) holds.
The postulates (P7)-(P10) follow directly (using the completeness theorem and taking into account (3)) from the conditions the conditions 1-4 of the definition of selection function, and (P12) follows from the fourth condition as well. Finally, let us prove (P11). Let $(\psi, I) *_t (\varphi, i) = (\psi', I')$ and $(\overline{\psi}, \overline{I}) *_t (\overline{\varphi}, \overline{i}) = (\overline{\psi}', \overline{I}')$, and suppose that $I = \overline{I}$, $i = \overline{i}$, and $\psi' \equiv \overline{\psi}'$. Then $Mod(\psi') = Mod(\overline{\psi}')$, i.e. $\min(Mod(\varphi), \leq^t_\psi) = \min(Mod(\varphi'), \leq^t_{\psi'})$, so

$$I' = \gamma^t_I(\min(Mod(\varphi), \leq^t_\psi), i) = \gamma^t_I(\min(Mod(\varphi'), \leq^t_{\psi'}), i) = \overline{I}'.$$

Finally, we briefly discuss iterated revision. It turns out to be straightforward to formulate the for iterated revision in our framework for the strong beliefs. Following DP, instead of performing revision on a strong belief formula, we have to perform revision on an abstract object called an *epistemic state* Ψ, which contains, in addition to the strong beliefs $Bel(\Psi)$, the entire information needed for coherent reasoning. We refer to an agent (Ψ, C) as an *epistemic state agent*.

Definition 6 (Iterated Agent Revision). *An iterated agent revision $*_t$ maps an epistemic state agent (Ψ, I), a strong belief $\varphi \in \mathbb{B}^{|t}$ and an intention i to an intention database I' such that if,*
$((\Psi, I) *_t (\varphi', i)) *_t (\varphi, c') = (\Psi', I')$
$(\Psi, I) *_t (\varphi, i) = (\overline{\Psi}', \overline{I}')$,
then following postulates hold:
(C1) If $\varphi \models \varphi'$, then $\Psi' \equiv \overline{\Psi'}$.
(C2) If $\varphi \models \neg\varphi'$, then $\Psi' \equiv \overline{\Psi'}$.
(C3) If $\overline{\Psi}' \models \varphi'$, then $\Psi' \models \varphi'$.
(C4) If $\overline{\Psi} \not\models \neg\varphi'$, then $\overline{\Psi}' \not\models \neg\varphi'$.

Theorem 3. *Suppose that an agent revision operator on epistemic states satisfies postulates (P1)-(P12). The operator satisfies postulates (C1)-(C4) iff when $(\Psi, I) *_t (\varphi, i) = (\Psi', I')$, the operator and its corresponding faithful assignment satisfy:*
(CR1) If $m_1 \models \varphi$ and $m_2 \models \varphi$, then $m_1 \leq^t_\Psi m_2$ iff $m_1 \leq^t_{\Psi'} m_2$.
(CR2) If $m_1 \not\models \varphi$ and $m_2 \not\models \varphi$, then $m_1 \leq^t_\Psi m_2$ iff $m_1 \leq^t_{\Psi'} m_2$.
(CR3) If $m_1 \models \varphi$, $m_2 \not\models \varphi$ and $m_1 <^t_\Psi m_2$, then $m_1 <^t_{\Psi'} m_2$.
(CR4) If $m_1 \models \varphi$, $m_2 \not\models \varphi$ and $m_1 \leq^t_\Psi m_2$, then $m_1 \leq^t_{\Psi'} m_2$.

Proof. Direct generalization of the proof of Darwiche and Pearl (using the same abbreviation $\circ_t$ as in the previous proof).

References

1. Marc van Zee, Dragan Doder, Mehdi Dastani, and Leendert van der Torre. AGM Revision of Beliefs about Action and Time. In *Proceedings of the International Joint Conference on Artificial Intelligence*, 2015.